4.5b Routh-Hurwitz criteria

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Jury conditions

Recall: For difference equations, local stability $\iff$ all eigenvalues of the Jacobian matrix had modulus <1. $|\lambda_i|<1$.

For ODEs, "local stability" $\iff$ all roots of the polynomial have negative real parts (or are negative).

Routh-Hurwitz criteria.

Def. (Hurwitz matrix)

Given the polynomial $P(\lambda) = \underbrace{\lambda^n}_{= a_0\lambda^n,\ a_0=1} + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n \underbrace{}_{+a_{n+1}\lambda^{-1},\ =0}$, $a_i \in \mathbb{R}$, the nth Hurwitz matrix of the polynomial is

$$\begin{cases} a_0 = 1 \\ a_i = 0 & \text{for } i<0 \\ a_i = 0 & \text{for } i>n \end{cases}$$

$$H_n = \begin{bmatrix} a_1 & a_0 \,(=1) & a_{-1}\,(=0) & a_{-2}\,(=0) & \cdots & a_{-n+2}\,(=0) \\ a_3 & a_2 & a_1 & a_0 & \cdots & a_{-n+4} \\ a_5 & a_4 & a_3 & a_2 & \cdots & a_{-n+6} \\ \vdots & & & \ddots & & \vdots \\ a_{2n-1}\,(=0) & a_{2n-2}\,(=0) & \cdots & & & a_n \end{bmatrix}$$

And the n Hurwitz matrices for $k<n$ are given by the principal minors of H_n. i.e.

$$H_1 = [a_1], \quad H_2 = \begin{bmatrix} a_1 & a_0 \\ a_3 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} \qquad H_3 = \begin{bmatrix} a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & \cdots & & & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & a_n \end{bmatrix}$$

Thm 4.4 (Routh-Hurwitz criteria)

All the roots of the polynomial $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$

All the roots of the polynomial $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ have negative real parts iff $\det(H_j) > 0$ for $j = 1, 2, \ldots, n$.

proof. Beyond scope of course.

proof for $n=2$: $P(\lambda) = \lambda^2 + a_1 \lambda + a_2 = 0$

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \qquad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}.$$

Routh-Hurwitz criteria is $\det(H_1) = \det(a_1) > 0 \implies \boxed{a_1 > 0}$

$$\det(H_2) = \det\begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix} = \det\begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$$

$$\implies \boxed{a_2 > 0.}$$

Forward case: Let $a_1 > 0$, $a_2 > 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}$. Then $Re(\lambda_1) = Re(\lambda_2) = \frac{-a_1}{2} < 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $a_1^2 - 4a_2 \geq 0$.

Clearly, $\lambda_2 < 0$ since both $-a_1 < 0$ and $-\sqrt{a_1^2 - 4a_2} \leq 0$.

Also, $a_1^2 - 4a_2 < a_1^2$ since $a_2 > 0$.

$$\implies \sqrt{a_1^2 - 4a_2} < a_1 \implies -a_1 + \sqrt{a_1^2 - 4a_2} < 0$$

$$\implies \lambda_1 < 0.$$

Backward case: Let $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$.

Suppose $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}_2$. Then $Re(\lambda_1) = Re(\lambda_2) = \frac{-a_1}{2} \implies a_1 > 0.$

Also $a_1^2 - 4a_2 < 0 \implies 4a_2 > a_1^2 > 0 \implies a_2 > 0.$

Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\lambda_1, \lambda_2 < 0$ and $a_1^2 - 4a_2 \geq 0$.

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} < 0 \implies \frac{-a_1}{2} < 0 \implies a_1 > 0.$$

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} < 0 \implies \frac{-a_1}{2} < 0 \implies a_1 > 0.$$

$$\implies -a_1 + \sqrt{a_1^2 - 4a_2} < 0$$

$$\implies \sqrt{a_1^2 - 4a_2} < a_1$$

$$\implies a_1^2 - 4a_2 < a_1^2$$

$$\implies -4a_2 < 0 \implies a_2 > 0.$$

$\boxed{\diagup}$

Ex. $n=2$, $a_1 > 0$, $a_2 > 0$

$n=3$, $a_1 > 0$, $a_3 > 0$, $a_1 a_2 > a_3$

$n=4$, $a_1 > 0$, $a_3 > 0$, $a_4 > 0$, $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$

$n=5$,
$$\begin{cases} a_i > 0, \quad i = 1,2,3,4,5 \\ a_1 a_2 a_3 > a_3^2 + a_1^2 a_4 \\ (a_1 a_4 - a_5)(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) > a_5 (a_1 a_2 - a_3)^2 + a_1 a_5^2. \end{cases}$$

Corollary 4.1 Given $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $a_i \in \mathbb{R}$, if all of the roots of $P(\lambda)$ have negative real parts, then $a_i > 0$ for $i = 1, 2, \ldots, n$.

Corollary of Thm 4.4. (Necessary, but not sufficient)

Direct proof. Let $-r_1, -r_2, \ldots, -r_{k_1}$ be the real roots of $P(\lambda)$, counting multiplicity.

Let $-c_1 \pm i d_1, \ldots, -c_{k_2} \pm i d_{k_2}$ be the complex roots, counting multiplicity.

$c_j, d_j, r_j \in \mathbb{R}$. Note that $r_j, c_j > 0$ by assumption.

Then
$$P(\lambda) = (\lambda + r_1) \cdots (\lambda + r_{k_1}) \left[(\lambda + c_1 - i d_1)(\lambda + c_1 + i d_1)\right] \cdots \left[(\lambda + c_{k_2} - i d_{k_2})(\lambda + c_{k_2} + i d_{k_2})\right]$$
$$= (\lambda + r_1) \cdots (\lambda + r_{k_1})(\lambda^2 + 2c_1 \lambda + c_1^2 + d_1^2) \cdots (\lambda^2 + 2c_{k_2}\lambda + c_{k_2}^2 + d_{k_2}^2).$$

$$= (\lambda+r_1)\cdots(\lambda+r_{k_1})(\lambda^2+2c_1\lambda+c_1^2+d_1^2)\cdots(\lambda^2+2c_{k_2}\lambda+c_{k_2}^2+d_{k_2}^2).$$

Note that all coefficients in factored equation are positive (>0), so after multiplying out, $a_1, \dots, a_n > 0$.

Ex. 4.6 $\quad \dfrac{d^3x}{dt^3} + a_2 \dfrac{dx}{dt} + a_3 x = 0, \qquad a_2, a_3 > 0.$

Char eq. $\quad P(\lambda) = \lambda^3 + a_2\lambda + a_3 = 0$, so $a_1 = 0 \not> 0$.

By Cor. 4.1, at least 1 root does not have pos. real part.

Ex. 4.7 $\quad \dddot{x} + 4\ddot{x} + \dot{x} + ax = 0$

$$P(\lambda) = \lambda^3 + 4\lambda^2 + \lambda + a.$$

By Thm 4.3, need all roots to have neg real part for sol. to approach 0.

By Routh-Hurwitz criteria, need

$$a_1 > 0, \quad a_3 > 0, \quad a_1 a_2 > a_3$$

Since $a_1 = 4$, $a_2 = 1$, $a_3 = a$ $\Rightarrow$ $a > 0$, $4 > a$

$\Rightarrow$ $4 > a > 0$ for sol. to approach 0.